A GLOBAL APPROACH TO THE THEORY OF CONNECTIONS IN FINSLER GEOMETRY

Nabil L. Youssef[†], **S. H. Abed**[†] **and A. Soleiman**[‡]

[†]Department of Mathematics, Faculty of Science,
Cairo University, Giza, Egypt
nlyoussef2003@yahoo.fr, sabed52@yahoo.fr

and

[‡]Department of Mathematics, Faculty of Science,
Benha University, Benha, Egypt
soleiman@mailer.eun.eg

Abstract. Adopting the pullback approach to Finsler geometry, the aim of the present paper is to provide intrinsic (coordinate-free) proofs of the existence and uniqueness theorems for the Chern (Rund) and Hashiguchi connections on a Finsler manifold. To accomplish this, we introduce and investigate the notions of semispray and nonlinear connection associated with a given regular connection, in the pullback bundle. Moreover, it is shown that for the the Chern (Rund) and Hashiguchi connections, the associated semispray coincides with the canonical spray and the associated nonlinear connection coincides with the Barthel connection. Explicit intrinsic expressions relating these connections and the Cartan connection are deduced.

Although our investigation is entirely global, the local expressions of the obtained results, when calculated, coincide with the existing classical local results.

We provide, for the sake of completeness and for comparison reasons, two appendices, one of them presenting a global survey of canonical linear connections in Finsler geometry and the other presenting a local survey of our global approach.

Keywords: Pullback bundle, π-vector field, Semispray, Nonlinear connection, Barthel connection, Regular connection, Cartan connection, Berwald connection, Chern connection, Hashiguchi connection.

AMS Subject Classification. 53C60, 53B40

Introduction

The most well-known and widely used approaches to GLOBAL Finsler geometry are the Klein-Grifone (KG-) approach (cf. [4], [5], [7]) and the pullback (PB-) approach (cf. [1], [3], [8], [12]). The universe of the first approach is the tangent bundle of $\mathcal{T}M$ (i.e, $\pi_{\mathcal{T}M} : T\mathcal{T}M \longrightarrow \mathcal{T}M$), whereas the universe of the second is the pullback of the tangent bundle TM by $\pi : \mathcal{T}M \longrightarrow M$ (i.e., $P : \pi^{-1}(TM) \longrightarrow \mathcal{T}M$). Each of the two approaches has its own geometry which differs significantly from the geometry of the other (in spite of the existence of some links between them).

The theory of connections is an important field of differential geometry. It was initially developed to solve pure geometrical problems. The most important linear connections in Finsler geometry were studied **locally** in [2], [9], [10], [11],...etc.

In [13], we have introduced and investigated new intrinsic proofs of intrinsic versions of the existence and uniqueness theorems for the Cartan and Berwald connections on a Finsler manifold (M, L). On the other hand, there are other connections of particular important in Finsler geometry, namely Chern and Hashiguchi connections. To the best of our knowledge, there is no proof of the existence and uniqueness theorems for the Chern and Hashiguchi connections from a purely global perspective.

The main purpose of the present paper is to provide **intrinsic** (coordinate-free) proofs of the existence and uniqueness theorems for the Chern and Hashiguchi connections within the pullback formalism, making simultaneous use of some concepts and results from the KG-approach. These proofs have the advantages of being simple, systematic and parallel to and guided by the Reimannian case.

The paper consists of three parts preceded by an introductory section (§1), which provides a brief account of the basic definitions and concepts necessary for this work. For more details, we refer to [12], [3], [4] and [5].

In the first part (§2), we review the fundamental results concerning the existence and uniqueness theorems for the Cartan and Berwald connections on a Finsler manifold (M, L) [13]. From these results, the relationships between the curvature tensors associated with the Berwald connection D° and the Cartan connection ∇ are obtained.

The second part (§3) is devoted to an intrinsic proof of the existence and uniqueness theorem of the Chern connection on a Finsler manifold (M, L) (Theorem 3.6). For the Chern connection, it is shown that the associated nonlinear connection coincides with the Barthel connection (Theorem 3.1). This establishes an important link between the PB-approach and the KG-approach. Moreover, the relationship between this connection and the Cartan connection is obtained (Theorem 3.7).

The third and last part (§4) provides an intrinsic proof of the existence and uniqueness theorem of the Hashiguchi connection on (M, L) (Theorem 4.3). The associated nonlinear connection is shown to coincide with the Barthel connection (Theorem 4.1). As in the previous section, the relationship between this connection and the Cartan connection is obtained (Theorem 4.4).

We have to emphasize that without the insertion of the KG-approach, we would have been unable to achieve these results. It should also be pointed out that the present work is formulated in a prospective modern coordinate-free form; the local expressions of the obtained results, when calculated, coincide with the existing classical local results.

Finally, for the sake of completeness and for comparison reasons, we provide two appendices, one of them presenting a global survey of canonical linear connections in Finsler geometry and the other presenting a local survey of our global approach. .

1. Notation and Preliminaries

In this section, we give a brief account of the basic concepts of the pullback formalism necessary for this work. For more details, we refer to [1], [3], [8] and [12]. We make the assumption that the geometric objects we consider are of class C^∞.
The following notation will be used throughout this paper:
M: a paracompact real differentiable manifold of finite dimension n and of class C^∞,
$\mathfrak{F}(M)$: the $\mathbb{R}$-algebra of differentiable functions on M,
$\mathfrak{X}(M)$: the $\mathfrak{F}(M)$-module of vector fields on M,
$\pi_M : TM \longrightarrow M$: the tangent bundle of M,
$\pi : \mathcal{T}M \longrightarrow M$: the subbundle of nonzero vectors tangent to M,
$V(TM)$: the vertical subbundle of the bundle TTM,
$P : \pi^{-1}(TM) \longrightarrow \mathcal{T}M$: the pullback of the tangent bundle TM by π,
$\mathfrak{X}(\pi(M))$: the $\mathfrak{F}(\mathcal{T}M)$-module of differentiable sections of $\pi^{-1}(TM)$,
i_X : the interior product with respect to $X \in \mathfrak{X}(M)$,
df : the exterior derivative of f,
$d_L := [i_L, d]$, i_L being the interior derivative with respect to a vector form L.

Elements of $\mathfrak{X}(\pi(M))$ will be called π-vector fields and will be denoted by barred letters $\overline{X}$. Tensor fields on $\pi^{-1}(TM)$ will be called π-tensor fields. The fundamental π-vector field is the π-vector field $\overline{\eta}$ defined by $\overline{\eta}(u) = (u,u)$ for all $u \in \mathcal{T}M$.

We have the following short exact sequence of vector bundles, relating the tangent bundle $T(\mathcal{T}M)$ and the pullback bundle $\pi^{-1}(TM)$:

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} T(\mathcal{T}M) \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

where the bundle morphisms ρ and γ are defined respectively by $\rho := (\pi_{\mathcal{T}M}, d\pi)$ and $\gamma(u,v) := j_u(v)$, where j_u is the natural isomorphism $j_u : T_{\pi_M(v)}M \longrightarrow T_u(T_{\pi_M(v)}M)$. The vector 1-form J on TM defined by $J := \gamma \circ \rho$ is called the natural almost tangent structure of TM. The vertical vector field $\mathcal{C}$ on TM defined by $\mathcal{C} := \gamma \circ \overline{\eta}$ is called the fundamental or the canonical (Liouville) vector field.

Let D be a linear connection (or simply a connection) on the pullback bundle $\pi^{-1}(TM)$. We associate with D the map

$$K : T\mathcal{T}M \longrightarrow \pi^{-1}(TM) : X \longmapsto D_X\overline{\eta},$$

called the connection (or the deflection) map of D. A tangent vector $X \in T_u(\mathcal{T}M)$ is said to be horizontal if $K(X) = 0$. The vector space $H_u(\mathcal{T}M) = \{X \in T_u(\mathcal{T}M) : K(X) = 0\}$ of the horizontal vectors at $u \in \mathcal{T}M$ is called the horizontal space to M at u . The connection D is said to be regular if

$$T_u(\mathcal{T}M) = V_u(\mathcal{T}M) \oplus H_u(\mathcal{T}M) \qquad \forall u \in \mathcal{T}M. \tag{1.1}$$

If M is endowed with a regular connection, then the vector bundle maps

$$\begin{aligned} \gamma &: \pi^{-1}(TM) \longrightarrow V(\mathcal{T}M), \\ \rho|_{H(\mathcal{T}M)} &: H(\mathcal{T}M) \longrightarrow \pi^{-1}(TM), \\ K|_{V(\mathcal{T}M)} &: V(\mathcal{T}M) \longrightarrow \pi^{-1}(TM) \end{aligned}$$

are vector bundle isomorphisms. Let us denote $\beta := (\rho|_{H(\mathcal{T}M)})^{-1}$, then

$$\rho \circ \beta = id_{\pi^{-1}(\mathcal{T}M)}, \qquad \beta \circ \rho = \begin{cases} id_{H(\mathcal{T}M)} & on\ H(\mathcal{T}M) \\ 0 & on\ V(\mathcal{T}M) \end{cases} \tag{1.2}$$

The map β will be called the horizontal map of the connection D.

According to the direct sum decomposition (1.1), a regular connection D gives rise to a horizontal projector h_D and a vertical projector v_D, given by

$$h_D = \beta \circ \rho, \qquad v_D = I - \beta \circ \rho, \tag{1.3}$$

where I is the identity endomorphism on $T(TM)$: $I = id_{T(TM)}$.

The (classical) torsion tensor $\mathbf{T}$ of the connection D is defined by

$$\mathbf{T}(X, Y) = D_X \rho Y - D_Y \rho X - \rho[X, Y] \quad \forall\, X, Y \in \mathfrak{X}(\mathcal{T}M).$$

The horizontal ((h)h-) and mixed ((h)hv-) torsion tensors, denoted by Q and T respectively, are defined by

$$Q(\overline{X}, \overline{Y}) = \mathbf{T}(\beta\overline{X}\beta\overline{Y}), \;\; T(\overline{X}, \overline{Y}) = \mathbf{T}(\gamma\overline{X}, \beta\overline{Y}) \quad \forall\, \overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M)).$$

If M is endowed with a metric g on $\pi^{-1}(TM)$, we write

$$T(\overline{X}, \overline{Y}, \overline{Z}) := g(T(\overline{X}, \overline{Y}), \overline{Z}). \tag{1.4}$$

The (classical) curvature tensor $\mathbf{K}$ of the connection D is defined by

$$\mathbf{K}(X, Y)\rho Z = -D_X D_Y \rho Z + D_Y D_X \rho Z + D_{[X,Y]}\rho Z \quad \forall\, X, Y, Z \in \mathfrak{X}(\mathcal{T}M).$$

The horizontal (h-), mixed (hv-) and vertical (v-) curvature tensors, denoted by R, P and S respectively, are defined by

$$R(\overline{X}, \overline{Y})\overline{Z} = \mathbf{K}(\beta\overline{X}\beta\overline{Y})\overline{Z}, \quad P(\overline{X}, \overline{Y})\overline{Z} = \mathbf{K}(\beta\overline{X}, \gamma\overline{Y})\overline{Z}, \quad S(\overline{X}, \overline{Y})\overline{Z} = \mathbf{K}(\gamma\overline{X}, \gamma\overline{Y})\overline{Z}.$$

The contracted curvature tensors, denoted by $\widehat{R}$, $\widehat{P}$ and $\widehat{S}$ respectively, are also known as the (v)h-, (v)hv- and (v)v-torsion tensors and are defined by

$$\widehat{R}(\overline{X}, \overline{Y}) = R(\overline{X}, \overline{Y})\overline{\eta}, \quad \widehat{P}(\overline{X}, \overline{Y}) = P(\overline{X}, \overline{Y})\overline{\eta}, \quad \widehat{S}(\overline{X}, \overline{Y}) = S(\overline{X}, \overline{Y})\overline{\eta}.$$

If M is endowed with a metric g on $\pi^{-1}(TM)$, we write

$$R(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) := g(R(\overline{X}, \overline{Y})\overline{Z}, \overline{W}), \; \cdots, \; S(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) := g(S(\overline{X}, \overline{Y})\overline{Z}, \overline{W}). \tag{1.5}$$

The following lemma is useful for subsequent use.

Lemma 1.1. *For every linear connection D on $\pi^{-1}(TM)$ with (classical) torsion $\boldsymbol{T}$ and (classical) curvature $\boldsymbol{K}$, we have*

(a) $\mathfrak{S}_{X,Y,Z}\{\boldsymbol{K}(X, Y)\rho Z + D_X \boldsymbol{T}(Y, Z) + \boldsymbol{T}(X, [Y, Z])\} = 0,$

(b) $\mathfrak{S}_{X,Y,Z}\{D_X \boldsymbol{K}(Y, Z) - \boldsymbol{K}(X, Y)D_Z - \boldsymbol{K}([X, Y], Z)\} = 0,$

where $\mathfrak{S}_{X,Y,Z}$ denotes cyclic sum over the vector fields X, Y and Z.

We terminate this section by some concepts and results concerning the Klein-Grifone approach. For more details, we refer to [4], [5] and [7].

A semispray on M is a vector field X on TM, C^∞ on $\mathcal{T}M$, C^1 on TM, such that $\rho \circ X = \overline{\eta}$. A semispray X which is homogeneous of degree 2 in the directional argument ($[\mathcal{C}, X] = X$) is called a spray.

Proposition 1.2. [7] *Let (M, L) be a Finsler manifold. The vector field G on TM defined by $i_G\Omega = -dE$ is a spray, where $E := \frac{1}{2}L^2$ is the energy function and $\Omega := dd_JE$. Such a spray is called the canonical spray.*

A nonlinear connection on M is a vector 1-form Γ on TM, C^∞ on $\mathcal{T}M$, C^0 on TM, such that

$$J\Gamma = J, \qquad \Gamma J = -J.$$

The horizontal and vertical projectors h_Γ and v_Γ associated with Γ are defined by $h_\Gamma := \frac{1}{2}(I + \Gamma)$, $v_\Gamma := \frac{1}{2}(I - \Gamma)$. Thus Γ gives rise to the direct sum decomposition $T\mathcal{T}M = H(\mathcal{T}M) \oplus V(\mathcal{T}M)$, where $H(\mathcal{T}M) := Im\, h_\Gamma = Ker\, v_\Gamma$, $V(\mathcal{T}M) := Im\, v_\Gamma = Ker\, h_\Gamma$. We have $J \circ h_\Gamma = J$, $h_\Gamma \circ J = 0$, $J \circ v_\Gamma = 0$, $v_\Gamma \circ J = J$. A nonlinear connection Γ is homogeneous if $[\mathcal{C}, \Gamma] = 0$. The torsion t of a nonlinear connection Γ is the vector 2-form on TM defined by $t := \frac{1}{2}[J, \Gamma]$. The curvature of Γ is the vector 2-form on TM defined by $\mathfrak{R} := -\frac{1}{2}[h_\Gamma, h_\Gamma]$. A nonlinear connection Γ is said to be conservative if $d_{h_\Gamma} E = 0$. With any given nonlinear connection Γ, one can associate a semispray S which is horizontal with respect to Γ, namely, $S = h_\Gamma S'$, where S' is an arbitrary semispray. Moreover, if Γ is homogeneous, then its associated semispray is a spray.

Theorem 1.3. [5] *On a Finsler manifold (M, L), there exists a unique conservative homogenous nonlinear connection with zero torsion. It is given by*:

$$\Gamma = [J, G],$$

where G is the canonical spray.
Such a nonlinear connection is called the canonical connection, or the Barthel connection, associated with (M, L).

It should be noted that the semispray associated with the Barthel connection is a spray, which is the canonical spray.

2. Cartan and Berwald connections in PB-formalism

In this section, we review the main results obtained in [13]. It concerns the spray and nonlinear connection associated with a regular connection on $\pi^{-1}(TM)$ and the existence and uniqueness theorems of Cartan and Berwald connections in Finsler geometry. From these results, the relationships between the curvature tensors associated with Berwald and Cartan connections are investigated.

Definition 2.1. *Let D be a regular connection on $\pi^{-1}(TM)$ with horizontal map β.*
– The semispray $S = \beta \circ \overline{\eta}$ will be called the semispray associated with D.
– The nonlinear connection $\Gamma = 2\beta \circ \rho - I$ will be called the nonlinear connection associated with D.

Proposition 2.2. *Let (M, L) be a Finsler manifold. Let D be a regular connection on $\pi^{-1}(TM)$ whose connection map is K and whose horizontal map is β. Then, the following assertions are equivalent:*

(a) *The (h)hv-torsion T of D has the property that $T(\overline{X}, \overline{\eta}) = 0$,*

(b) $K = \gamma^{-1}$ *on $V(TM)$,*

(c) $\widetilde{\Gamma} := \beta \circ \rho - \gamma \circ K$ *is a nonlinear connection on M.*

Consequently, if any one of the above assertions holds, then $\widetilde{\Gamma}$ coincides with the nonlinear connection associated with D: $\widetilde{\Gamma} = \Gamma = 2\beta \circ \rho - I$, and in this case $h_\Gamma = h_D = \beta \circ \rho$ and $v_\Gamma = v_D = \gamma \circ K$.

Lemma 2.3. *Let D be a regular connection on $\pi^{-1}(TM)$ whose (h)hv-torsion tensor T has the property that $T(\overline{X}, \overline{\eta}) = 0$. Then, we have:*

(a) $[\beta\overline{X}, \beta\overline{Y}] = \gamma\widehat{R}(\overline{X}, \overline{Y}) + \beta(D_{\beta\overline{X}}\overline{Y} - D_{\beta\overline{Y}}\overline{X} - Q(\overline{X}, \overline{Y}))$,

(b) $[\gamma\overline{X}, \beta\overline{Y}] = -\gamma(\widehat{P}(\overline{Y}, \overline{X}) + D_{\beta\overline{Y}}\overline{X}) + \beta(D_{\gamma\overline{X}}\overline{Y} - T(\overline{X}, \overline{Y}))$,

(c) $[\gamma\overline{X}, \gamma\overline{Y}] = \gamma(D_{\gamma\overline{X}}\overline{Y} - D_{\gamma\overline{Y}}\overline{X} + \widehat{S}(\overline{X}, \overline{Y}))$.

The following theorem guarantees the existence and uniqueness of the Cartan connection. This is the Finsler analogue of the fundamental theorem of Riemannian geometry.

Theorem 2.4. *Let (M, L) be a Finsler manifold and g the Finsler metric defined by L. There exists a unique regular connection ∇ on $\pi^{-1}(TM)$ such that*

(a) *∇ is metric*: $\nabla g = 0$,

(b) *The (h)h-torsion of ∇ vanishes*: $Q = 0$,

(c) *The (h)hv-torsion T of ∇ satisfies*: $g(T(\overline{X}, \overline{Y}), \overline{Z}) = g(T(\overline{X}, \overline{Z}), \overline{Y})$.

Such a connection is called the Cartan connection associated with the Finsler manifold (M, L).

This connection is uniquely determined by the relations:

(i) $2g(\nabla_{\gamma\overline{X}}\overline{Y}, \overline{Z}) = \gamma\overline{X} \cdot g(\overline{Y}, \overline{Z}) + g(\overline{Y}, \rho[\beta\overline{Z}, \gamma\overline{X}]) + g(\overline{Z}, \rho[\gamma\overline{X}, \beta\overline{Y}])$.

(ii)
$$\begin{aligned} 2g(\nabla_{\beta\overline{X}}\rho Y, \rho Z) = {}& \beta\overline{X} \cdot g(\overline{Y}, \overline{Z}) + \beta\overline{Y} \cdot g(\overline{Z}, \overline{X}) - \beta\overline{Z} \cdot g(\overline{X}, \overline{Y}) \\ & - g(\overline{X}, \rho[\beta\overline{Y}, \beta\overline{Z}]) + g(\overline{Y}, \rho[\beta\overline{Z}, \beta\overline{X}]) + g(\overline{Z}, \rho[\beta\overline{X}, \beta\overline{Y}]). \end{aligned}$$

Concerning the nonlinear connection associated with the Cartan connection, we have :

Theorem 2.5. *Let ∇ be the Cartan connection. The nonlinear connection Γ associated with ∇ coincides with the Barthel connection*: $\Gamma = [J, G]$.

Concerning the existence and uniqueness of the Berwald connection, we have:

Theorem 2.6. *Let (M, L) be a Finsler manifold. There exists a unique regular connection D° on $\pi^{-1}(TM)$ such that*

(a) $D^\circ_{h^\circ X} L = 0$,

(b) *D° is torsion-free:* $\boldsymbol{T}^\circ = 0$,

(c) *The (v)hv-torsion tensor* $\widehat{P^\circ}$ *of* D° *vanishes:* $\widehat{P^\circ}(\overline{X}, \overline{Y}) = 0$.

Such a connection is called the Berwald connection associated with the Finsler manifold (M, L).

Proposition 2.7. *The semispray associated with the Berwald connection is a spray which coincides with the canonical spray. Moreover, the nonlinear connection associated with the Berwald connection coincides with the Barthel connection.*

Theorem 2.8. *The Berwald connection D° is explicitly expressed in terms of the Cartan connection ∇ in the form:*

$$D^\circ_X \overline{Y} = \nabla_X \overline{Y} + \widehat{P}(\rho X, \overline{Y}) - T(KX, \overline{Y}). \tag{2.1}$$

In particular, we have

(a) $D^\circ_{\gamma\overline{X}} \overline{Y} = \nabla_{\gamma\overline{X}} \overline{Y} - T(\overline{X}, \overline{Y})$.

(b) $D^\circ_{\beta\overline{X}} \overline{Y} = \nabla_{\beta\overline{X}} \overline{Y} + \widehat{P}(\overline{X}, \overline{Y})$.

In view of the above theorem, we have:

Proposition 2.9. *The Berwald connection D° has the properties:*

(a) $(D^\circ_{\gamma\overline{X}}\, g)(\overline{Y}, \overline{Z}) = 2g(T(\overline{X}, \overline{Y}), \overline{Z})$.

(b) $(D^\circ_{\beta\overline{X}}\, g)(\overline{Y}, \overline{Z}) = -2g(\widehat{P}(\overline{X}, \overline{Y}), \overline{Z})$.

(c) $D^\circ_G\, g = 0$.

Consequently,
– a Finsler manifold (M, L) is Riemannian if and only if $D^\circ_{\gamma\overline{X}}\, g = 0$.
– a Finsler manifold (M, L) is Landsbergian if and only if $D^\circ_{\beta\overline{X}}\, g = 0$.

Proposition 2.10. *The Curvature tensor $\boldsymbol{K}$ of the Cartan connection ∇ and The Curvature tensor $\boldsymbol{K}^\circ$ of the Berwald connection D° are related by:*

$$\begin{aligned}\boldsymbol{K}^\circ(X, Y)\overline{Z} = \boldsymbol{K}(X, Y)\overline{Z} &- \widehat{P}(\boldsymbol{T}(X, Y), \overline{Z}) - T(K[X, Y], \overline{Z}) - \mathfrak{U}_{X,Y}\{(\nabla_X \widehat{P})(\rho Y, \overline{Z}) \\ &- \nabla_X T(KY, \overline{Z}) + T(KY, \nabla_X \overline{Z}) + \widehat{P}(\rho X, \widehat{P}(\rho Y, \overline{Z})) - \widehat{P}(\rho X, T(KY, \overline{Z})) \\ &+ T(KX, T(KY, \overline{Z})) - T(KX, \widehat{P}(\rho Y, \overline{Z}))\},\end{aligned}$$

where $\mathfrak{U}_{X,Y} A(X, Y) = A(X, Y) - A(Y, X)$.

In particular, we have

(a) $S^\circ(\overline{X}, \overline{Y})\overline{Z} = 0$.

(b) $$\begin{aligned}P^\circ(\overline{X}, \overline{Y})\overline{Z} = P(\overline{X}, \overline{Y})\overline{Z} &+ (\nabla_{\gamma\overline{Y}} \widehat{P})(\overline{X}, \overline{Z}) + \widehat{P}(T(\overline{Y}, \overline{X}), \overline{Z}) + \widehat{P}(\overline{X}, T(\overline{Y}, \overline{Z})) + \\ &+ (\nabla_{\beta\overline{X}} T)(\overline{Y}, \overline{Z}) - T(\overline{Y}, \widehat{P}(\overline{X}, \overline{Z})) - T(\widehat{P}(\overline{X}, \overline{Y}), \overline{Z}).\end{aligned}$$

(c) $R^\circ(\overline{X}, \overline{Y})\overline{Z} = R(\overline{X}, \overline{Y})\overline{Z} - T(\widehat{R}(\overline{X}, \overline{Y}), \overline{Z}) - \mathfrak{U}_{\overline{X},\overline{Y}}\{(\nabla_{\beta\overline{X}} \widehat{P})(\overline{Y}, \overline{Z}) + \widehat{P}(\overline{X}, \widehat{P}(\overline{Y}, \overline{Z}))\}$.

Corollary 2.11.

(a) $\widehat{S^\circ}(\overline{X},\overline{Y})=0$,

(b) $\widehat{P^\circ}(\overline{X},\overline{Y})=0$,

(c) $\widehat{R^\circ}(\overline{X},\overline{Y})=\widehat{R}(\overline{X},\overline{Y})$.

3. Chern (Rund) connection

In this section, we establish an intrinsic (coordinate-free) proof of the existence and uniqueness theorem of Chern (Rund) connection. Moreover, the relationships between this connection and the Cartan and Berwald connections are obtained.

We start with the following fundamental result.

Theorem 3.1. *Let $D^\diamond$ be a regular connection on $\pi^{-1}(TM)$ with connection map $K^\diamond$ such that*

(a) $(D^\diamond_X g)(\rho Y,\rho Z)=2g(T(K^\diamond X,\rho Y),\rho Z)$,

(b) *$D^\diamond$ is torsion free*: $\boldsymbol{T}^\diamond=0$.

Then, the nonlinear connection $\Gamma^\diamond$ associated with $D^\diamond$ coincides with the Barthel connection: $\Gamma=[J,G]$.

To prove this theorem, we need the following three lemmas.968

Lemma 3.2. *The hv-curvature tensor $P^\diamond$ of the connection $D^\diamond$ is symmetric with respect to the first and third arguments*:

$$P^\diamond(\overline{X},\overline{Y})\overline{Z}=P^\diamond(\overline{Z},\overline{Y})\overline{X}\quad \textit{for all } \overline{X},\overline{Y},\overline{Z}\in\mathfrak{X}(\pi(M)).$$

Proof. The proof follows from Lemma 1.1(a) by setting $X=\beta^\diamond\,\overline{X}$, $Y=\gamma\overline{Y}$ and $Z=\beta^\diamond\,\overline{Z}$, noting that $\rho\circ\gamma=0$, $\rho\circ\beta^\diamond=id_{\mathfrak{X}(\pi(M))}$ and that $\mathbf{T}^\diamond=0$. $\square$

Lemma 3.3. *The hv-curvature tensor $P^\diamond$ of the connection $D^\diamond$ has the property that*

$$P^\diamond(\overline{X},\overline{Y},\overline{Z},\overline{W})+P^\diamond(\overline{X},\overline{Y},\overline{W},\overline{Z})=2(D^\diamond_{\beta^\diamond\,\overline{X}}T)(\overline{Y},\overline{Z},\overline{W})-2T(\widehat{P^\diamond}(\overline{X},\overline{Y}),\overline{Z},\overline{W}).$$

Proof. We have

$$X\cdot g(\overline{W},\overline{Z})=(D^\diamond_X g)(\overline{W},\overline{Z})+g(D^\diamond_X\overline{W},\overline{Z})+g(\overline{W},D^\diamond_X\overline{Z}).$$

From which, we obtain

$$\begin{aligned}
X\cdot(Y\cdot g(\overline{W},\overline{Z})) &= X\cdot((D^\diamond_Y g)(\overline{W},\overline{Z}))+X\cdot g(D^\diamond_Y\overline{W},\overline{Z})+X\cdot g(\overline{W},D^\diamond_Y\overline{Z})\\
&= X\cdot((D^\diamond_Y g)(\overline{W},\overline{Z}))+(D^\diamond_X g)(D^\diamond_Y\overline{W},\overline{Z})+(D^\diamond_X g)(\overline{W},D^\diamond_Y\overline{Z})\\
&\quad +g(D^\diamond_X D^\diamond_Y\overline{W},\overline{Z})+g(D^\diamond_Y\overline{W},D^\diamond_X\overline{Z})+g(D^\diamond_X\overline{W},D^\diamond_Y\overline{Z})+\\
&\quad +g(\overline{W},D^\diamond_X D^\diamond_Y\overline{Z}),
\end{aligned}$$

with similar expression for $Y \cdot (X \cdot g(\overline{W}, \overline{Z}))$. Then,

$$\begin{aligned}
[X, Y] \cdot g(\overline{W}, \overline{Z}) &= (D^{\diamond}_{[X,Y]} g)(\overline{W}, \overline{Z}) + g(D^{\diamond}_{[X,Y]} \overline{W}, \overline{Z}) + g(\overline{W}, D^{\diamond}_{[X,Y]} \overline{Z}) \\
&= \mathfrak{U}_{X,Y}\{X \cdot ((D^{\diamond}_{Y} g)(\overline{W}, \overline{Z})) + (D^{\diamond}_{X} g)(D^{\diamond}_{Y} \overline{W}, \overline{Z}) + (D^{\diamond}_{X} g)(\overline{W}, D^{\diamond}_{Y} \overline{Z})\} \\
&\quad + g([D^{\diamond}_{X}, D^{\diamond}_{Y}] \overline{W}, \overline{Z}) + g(\overline{W}, [D^{\diamond}_{X}, D^{\diamond}_{Y}] \overline{Z}).
\end{aligned}$$

Hence,

$$\begin{aligned}
& g(\mathbf{K}(X, Y)\overline{Z}, \overline{W}) + g(\mathbf{K}(X, Y)\overline{W}, \overline{Z}) = \\
&= 2\mathfrak{U}_{X,Y}\{X \cdot g(T(K^{\diamond} Y, \overline{W}), \overline{Z}) + g(T(K^{\diamond} X, D^{\diamond}_{Y} \overline{W}), \overline{Z}) \\
&\quad + g(T(K^{\diamond} X, \overline{W}), D^{\diamond}_{Y} \overline{Z})\} - 2g(T(K^{\diamond}[X, Y], \overline{W}), \overline{Z}).
\end{aligned} \tag{3.1}$$

From which, by setting $X = \beta^{\diamond}\, \overline{X}$ and $Y = \gamma \overline{Y}$ into (3.1) and using Lemma 2.3, we get

$$\begin{aligned}
P^{\diamond}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) + P^{\diamond}(\overline{X}, \overline{Y}, \overline{W}, \overline{Z}) &= 2\beta^{\diamond}\, \overline{X} \cdot T(\overline{Y}, \overline{W}, \overline{Z}) - 2T(\overline{Y}, D^{\diamond}_{\beta^{\diamond}\, \overline{X}} \overline{W}, \overline{Z}) \\
&\quad - 2T(\overline{Y}, \overline{W}, D^{\diamond}_{\beta^{\diamond}\, \overline{X}} \overline{Z}) - 2T(\widehat{P^{\diamond}}(\overline{X}, \overline{Y}), \overline{W}, \overline{Z}) - \\
&\quad - 2T(D^{\diamond}_{\beta^{\diamond}\, \overline{X}} \overline{Y}, \overline{W}, \overline{Z}).
\end{aligned}$$

Hence, the result follows. $\square$

Lemma 3.4. *The (v)hv-torsion tensor* $\widehat{P^{\diamond}}$ *given by:*

$$\widehat{P^{\diamond}}(\overline{X}, \overline{Y}) = (D^{\diamond}_{\beta^{\diamond}\, \overline{\eta}} T)(\overline{X}, \overline{Y}).$$

Consequently, $\widehat{P^{\diamond}}$ *is symmetric and* $\widehat{P^{\diamond}}(\overline{X}, \overline{\eta}) = 0$.

Proof. Firstly, one can easily show that

$$(D^{\diamond}_{\beta^{\diamond}\, \overline{X}} T)(\overline{Y}, \overline{Z}, \overline{W}) = g((D^{\diamond}_{\beta^{\diamond}\, \overline{X}} T)(\overline{Y}, \overline{Z}), \overline{W}). \tag{3.2}$$

Cyclic permutation on $\overline{X}, \overline{Z}, \overline{W}$ in the formula of Lemma 3.3 yields three equations. Adding two of these equations and subtracting the third gives

$$\begin{aligned}
P^{\diamond}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) &= (D_{\beta^{\diamond}\, \overline{X}} T)(\overline{Y}, \overline{Z}, \overline{W}) + (D_{\beta^{\diamond}\, \overline{Z}} T)(\overline{Y}, \overline{W}, \overline{X}) - (D_{\beta^{\diamond}\, \overline{W}} T)(\overline{Y}, \overline{X}, \overline{Z}) \\
&\quad + T(\widehat{P^{\diamond}}(\overline{W}, \overline{Y}), \overline{X}, \overline{Z}) - T(\widehat{P^{\diamond}}(\overline{X}, \overline{Y}), \overline{Z}, \overline{W}) - T(\widehat{P^{\diamond}}(\overline{Z}, \overline{Y}), \overline{W}, \overline{X}).
\end{aligned} \tag{3.3}$$

Setting $\overline{X} = \overline{Z} = \overline{\eta}$ in (3.3), taking into account the properties of the (h)hv-torsion T and the fact that $K^{\diamond} \circ \beta^{\diamond} = 0$, we obtain

$$\widehat{P^{\diamond}}(\eta, \overline{Y}) = 0, \ \forall\, \overline{Y} \in \mathfrak{X}(\pi(M)). \tag{3.4}$$

Again, setting $\overline{Z} = \overline{\eta}$ in (3.3) and now using (3.4) and (3.2), we conclude that

$$\widehat{P^{\diamond}}(\overline{X}, \overline{Y}) = (D^{\diamond}_{\beta^{\diamond}\, \overline{\eta}} T)(\overline{X}, \overline{Y}). \tag{3.5}$$

The symmetry of $\widehat{P^{\diamond}}$ follows then from (3.5) and the symmetry of T. $\square$

Proof of Theorem 3.1:

As the (h)hv-torsion tensor $T^{\diamond}$ of the connection $D^{\diamond}$ vanishes, then $T^{\diamond}(\overline{X}, \overline{\eta}) = 0$. Consequently, by Proposition 2.2, it follows that the associated nonlinear connection $\Gamma^{\diamond}$ has the form

$$\Gamma^{\diamond} := \beta^{\diamond}\, o\rho - \gamma o K^{\diamond}.$$

Now, we prove that the connection $\Gamma^\diamond$ satisfies the following properties:

$\Gamma^\diamond$ ***is conservative:*** $d_{h^\diamond X}E = 0$, for all $X \in \mathfrak{X}(\mathcal{T}M)$:
In fact, by condition (a), $D^\diamond_{h^\diamond X}g = 0$. Taking this, together with the identities $2E = g(\overline{\eta}, \overline{\eta})$ and $K^\diamond \circ h^\diamond = 0$ into account, we get

$$d_{h^\diamond X}E = h^\diamond X \cdot E = \frac{1}{2} h^\diamond X \cdot g(\overline{\eta}, \overline{\eta}) = g(D^\diamond_{h^\diamond X}\overline{\eta}, \overline{\eta}) = g(K^\diamond(h^\diamond X), \overline{\eta}) = 0.$$

$\Gamma^\diamond$ ***is homogenous*** ($[\mathcal{C}, \Gamma^\diamond] = 0$)***:***

It is easy to show that

$$[\mathcal{C}, v^\diamond]X = -v^\diamond[\mathcal{C}, h^\diamond X]$$

As $v^\diamond = \gamma \circ K^\diamond$, $h^\diamond = \beta^\diamond \circ \rho$ and $\gamma \circ \overline{\eta} = \mathcal{C}$, then

$$[\mathcal{C}, v^\diamond]X = -(\gamma \circ K^\diamond)[\gamma\overline{\eta}, \beta^\diamond \rho X]$$

Using Lemma 2.3, we obtain

$$\begin{aligned}[\mathcal{C}, v^\diamond]X &= -(\gamma \circ K^\diamond)\{-\gamma(\widehat{P^\diamond}(\rho X, \overline{\eta}) + D^\diamond_{\beta^\diamond \rho X}\overline{\eta}) + \beta^\diamond D^\diamond_{\gamma\overline{\eta}}\rho X\} \\ &= \gamma\widehat{P^\diamond}(\rho X, \overline{\eta}), \ \ as \ \ K^\diamond \circ \beta^\diamond = 0 \ \ and \ K^\diamond \circ \gamma = id_{\mathfrak{X}(\pi(M))}.\end{aligned}$$

Finally, by Lemma 3.4, $\widehat{P^\diamond}(\overline{X}, \overline{\eta}) = 0$ and consequently, $[\mathcal{C}, \Gamma^\diamond] = -2[\mathcal{C}, v^\diamond] = 0$.

$\Gamma^\diamond$ ***is torsion-free*** ($[J, \Gamma^\diamond] = 0$)***:***

$$\begin{aligned}[J, v^\diamond](X, Y) &= [JX, v^\diamond Y] + [v^\diamond X, JY] + v^\diamond J[X, Y] + Jv^\diamond[X, Y] \\ &\quad -J[v^\diamond X, Y] - J[X, v^\diamond Y] - v^\diamond[JX, Y] - v^\diamond[X, JY].\end{aligned}$$

As $J \circ v^\diamond = 0$, $v^\diamond \circ J = J$ and the vertical distribution is completely integrable, we get

$$\begin{aligned}[J, v^\diamond](X, Y) &= J[h^\diamond X, h^\diamond Y] - v^\diamond[JX, h^\diamond Y] - v^\diamond[h^\diamond X, JY] \\ &= J[\beta^\diamond \rho X, \beta^\diamond \rho Y] - v^\diamond[\gamma\rho X, \beta^\diamond \rho Y] + v^\diamond[\gamma\rho Y, \beta^\diamond \rho X].\end{aligned}$$

From which, together with Lemma 2.3, we obtain

$$\begin{aligned}[J, v^\diamond](X, Y) &= J\{\gamma\widehat{R^\diamond}(\rho X, \rho Y) + \beta^\diamond (D^\diamond_{h^\diamond X}\rho Y - D^\diamond_{h^\diamond Y}\rho X)\} \\ &\quad -(\gamma \circ K^\diamond)\{-\gamma(\widehat{P^\diamond}(\rho Y, \rho X) + D^\diamond_{hY}\rho X) + \beta^\diamond (D^\diamond_{JX}\rho Y)\} \\ &\quad +(\gamma \circ K^\diamond)\{-\gamma(\widehat{P^\diamond}(\rho X, \rho Y) + D^\diamond_{h^\diamond X}\rho Y) + \beta^\diamond (D^\diamond_{JY}\rho X)\}\end{aligned}$$

Noting that $J \circ \gamma = 0$, $J \circ \beta^\diamond = \gamma$, $K^\diamond \circ \gamma = id_{\mathfrak{X}(\pi(M))}$, $K^\diamond \circ \beta^\diamond = 0$ and that $\widehat{P^\diamond}$ is symmetric (by Lemma 3.4), it follows that $[J, v^\diamond] = 0$. From which $t := \frac{1}{2}[J, \Gamma^\diamond] = -[J, v^\diamond] = 0$.

From the above consideration, $\Gamma^\diamond = \beta^\diamond \circ \rho - \gamma \circ K^\diamond$ is a conservative torsion-free homogenous nonlinear connection. By the uniqueness of the Barthel connection (Theorem 1.3), it follows that $\Gamma^\diamond$ coincides with the Barthel connection $[J, G]$. $\square$

In view of Theorem 3.1, Theorem 2.5 and Proposition 2.7, we have:

Corollary 3.5. *The nonlinear connection associated with the connection $D^\diamond$ is the same as the nonlinear connection associated with the Cartan connection ∇ or the Berwald connection D°, which coincides with the Barthel connection $[J, G]$.*
Consequently, $h^\diamond = h^\circ = h = \beta \circ \rho$, $v^\diamond = v^\circ = v = \gamma \circ K$ and hence $\beta^\diamond = \beta^\circ = \beta$, $K^\diamond = K^\circ = K$.

Now, we are in a position to announce the main result of this section.

Theorem 3.6. *Let (M, L) be a Finsler manifold and g the Finsler metric defined by L. There exists a unique regular connection $D^\diamond$ on $\pi^{-1}(TM)$ such that*

(a) $(D^\diamond_X g)(\rho Y, \rho Z) = 2g(T(K^\diamond X, \rho Y), \rho Z)$,

(b) *$D^\diamond$ is torsion free*: $\boldsymbol{T}^\diamond = 0$,

where T is the (h)hv-torsion of the Cartan connection and $K^\diamond$ is the connection map of $D^\diamond$.
This connection is called the Chern connection associated with (M, L).

Proof. First we prove the ***uniqueness***. If $D^\diamond$ is a non-metric linear connection on $\pi^{-1}(TM)$ with nonzero torsion $\mathbf{T}^\diamond$, then $D^\diamond$ is completely determined by:

$$\left.\begin{aligned} 2g(D^\diamond_X \rho Y, \rho Z) = {} & X \cdot g(\rho Y, \rho Z) + Y \cdot g(\rho Z, \rho X) - Z \cdot g(\rho X, \rho Y) \\ & - g(\rho X, \mathbf{T}^\diamond(Y, Z)) + g(\rho Y, \mathbf{T}^\diamond(Z, X)) + g(\rho Z, \mathbf{T}^\diamond(X, Y)) \\ & - g(\rho X, \rho[Y, Z]) + g(\rho Y, \rho[Z, X]) + g(\rho Z, \rho[X, Y]) \\ & - (D^\diamond_X g)(\rho Y, \rho Z) - (D^\diamond_Y g)(\rho Z, \rho X) + (D^\diamond_Z g)(\rho X, \rho Y). \end{aligned}\right\} \tag{3.6}$$

for all $X, Y, Z \in \mathfrak{X}(\mathcal{T}M)$. The connection $D^\diamond$ being regular, let $h^\diamond$ and $v^\diamond$ be its horizontal and vertical projectors: $h^\diamond = \beta^\diamond \circ \rho$, $v^\diamond = I - \beta^\diamond \circ \rho$. From Corollary 3.5, we have $h^\diamond = h = \beta \circ \rho$ and $v^\diamond = v = \gamma \circ K$.

Now, by replacing X, Y, Z by hX, hY, hZ in (3.6) and using conditions (a) and (b), taking into account the fact that $\rho \circ h = \rho$, $K \circ h = 0$, we get

$$\begin{aligned} 2g(D^\diamond_{hX} \rho Y, \rho Z) = {} & hX \cdot g(\rho Y, \rho Z) + hY \cdot g(\rho Z, \rho X) - hZ \cdot g(\rho X, \rho Y) \\ & - g(\rho X, \rho[hY, hZ]) + g(\rho Y, \rho[hZ, hX]) + g(\rho Z, \rho[hX, hY]). \end{aligned}$$

From which, together with Theorem 2.4(ii), we get

$$D^\diamond_{hX} \rho Y = \nabla_{hX} \rho Y. \tag{3.7}$$

Similarly, by replacing X, Y, Z by vX, vY, vZ in (3.6), where $vX = \gamma \overline{X}$ for some $\overline{X} \in \mathfrak{X}(\pi(M))$ and using conditions (a) and (b), noting that $\rho \circ v = 0$, we have

$$\begin{aligned} 2g(D^\diamond_{vX} \rho Y, \rho Z) &= vX \cdot g(\rho Y, \rho Z) + g(\rho Y, \rho[hZ, vX]) + g(\rho Z, \rho[vX, hY]). \\ & \quad - (D^\diamond_{vX} g)(\rho Y, \rho Z) \\ &= \gamma \overline{X} \cdot g(\rho Y, \rho Z) + g(\rho Y, \rho[hZ, \gamma \overline{X}]) + g(\rho Z, \rho[\gamma \overline{X}, hY]). \\ & \quad - 2g(T(\overline{X}, \rho Y), \rho Z). \end{aligned}$$

From which, together with Theorem 2.4(i), we have

$$D^\diamond_{vX} \rho Y = \nabla_{vX} \rho Y - T(KX, \rho Y) = \rho[vX, hY]. \tag{3.8}$$

Now, from Equations (3.7) and (3.8), we get

$$D^{\diamond}_X\overline{Y} = \nabla_X\overline{Y} - T(KX, \overline{Y}). \tag{3.9}$$

Hence $D^{\diamond}_X\rho Y$ is uniquely determined by Equations (3.9).

To show the ***existence***, we define $D^{\diamond}$ by the requirement that (3.9) holds for all $X, Y \in \mathfrak{X}(\mathcal{T}M)$. Now, we need to prove that the connection $D^{\diamond}$ satisfies the following properties:

$D^{\diamond}$ ***satisfies condition (a)***: By using (3.9) and the properties of the Cartan connection ∇, we get

$$\begin{aligned}(D^{\diamond}_X\, g)(\overline{Y}, \overline{Z}) &= D^{\diamond}_X\, g(\overline{Y}, \overline{Z}) - g(D^{\diamond}_X\overline{Y}, \overline{Z}) - g(\overline{Y}, D^{\diamond}_X\overline{Z})\\ &= X \cdot g(\overline{Y}, \overline{Z}) - g(\nabla_X\overline{Y} - T(KX, \overline{Y}), \overline{Z}) - g(\overline{Y}, \nabla_X\overline{Z} - T(KX, \overline{Z}))\\ &= (\nabla_X g)(\overline{Y}, \overline{Z}) + g(T(KX, \overline{Y}), \overline{Z}) + g(\overline{Y}, T(KX, \overline{Z}))\\ &= 2g(T(KX, \overline{Y}), \overline{Z}) = 2g(T(K^{\diamond}X, \overline{Y}), \overline{Z}).\end{aligned}$$

$D^{\diamond}$ ***satisfies condition (b)***: Again using (3.9)and the properties of the ∇, we get

$$\begin{aligned}\mathbf{T}^{\diamond}(X, Y) &= D^{\diamond}_X\rho Y - D^{\diamond}_Y\rho X - \rho[X, Y]\\ &= \nabla_X\rho Y - T(KX, \rho Y) - \nabla_Y\rho X + T(KY, \rho X) - \rho[X, Y]\\ &= \mathbf{T}(X, Y) - \mathbf{T}(vX, hY) + \mathbf{T}(vY, hX) = 0.\end{aligned}$$

This completes the proof. □

It is to be noted that when we localize the above result, the *local expressions* obtained coincide with the classical expressions found in [2], [6], [10]...etc. (c.f. Appendix 2).

In view of the above theorem, taking Theorem 2.8 into account, we have

Theorem 3.7. *The Chern connection $D^{\diamond}$ is given in terms of the Cartan connection ∇ (or the Berwald connection D°) by:*

$$D^{\diamond}_X\overline{Y} = \nabla_X\overline{Y} - T(KX, \overline{Y}) = D^{\circ}_X\overline{Y} - \widehat{P}(\rho X, \overline{Y}).$$

In particular, we have

(a) $D^{\diamond}_{\gamma\overline{X}}\overline{Y} = \nabla_{\gamma\overline{X}}\overline{Y} - T(\overline{X}, \overline{Y}) = D^{\circ}_{\gamma\overline{X}}\overline{Y}$.

(b) $D^{\diamond}_{\beta\overline{X}}\overline{Y} = \nabla_{\beta\overline{X}}\overline{Y} = D^{\circ}_{\beta\overline{X}}\overline{Y} - \widehat{P}(\overline{X}, \overline{Y})$.

Corollary 3.8.

(a) $(D^{\diamond}_{\beta^{\diamond}\,\overline{X}}g)(\overline{Y}, \overline{Z}) = 0$.

(b) $(D^{\diamond}_{\gamma\overline{X}}g)(\overline{Y}, \overline{Z}) = 2g(T(\overline{X}, \overline{Y}), \overline{Z})$.

Consequently, Finsler manifold (M, L) is Riemannian if, and only if, $D^{\diamond}_{\gamma\overline{X}}g = 0$.

Concerning the curvature tensor of the Chern connection $D^{\diamond}$, we have:

Proposition 3.9. *The curvature tensor of the Chern connection $D^\diamond$ is given, in terms of the curvature tensor of the Cartan connection ∇, by:*

$$\begin{aligned}\boldsymbol{K}^\diamond(X,Y)\overline{Z} = \boldsymbol{K}(X,Y)\overline{Z} - T(K[X,Y],\overline{Z}) + \mathfrak{U}_{X,Y}\{&\nabla_X T(KY,\overline{Z})\\ &-T(KY,\nabla_X\overline{Z}) - T(KX, T(KY,\overline{Z}))\}.\end{aligned}$$

In particular, we have

(a) $S^\diamond(\overline{X},\overline{Y})\overline{Z} = 0$.

(b) $P^\diamond(\overline{X},\overline{Y})\overline{Z} = P(\overline{X},\overline{Y})\overline{Z} - T(\widehat{P}(\overline{X},\overline{Y}),\overline{Z}) + (\nabla_{\beta\overline{X}}T)(\overline{Y},\overline{Z})$.

(c) $R^\diamond(\overline{X},\overline{Y})\overline{Z} = R(\overline{X},\overline{Y})\overline{Z} - T(\widehat{R}(\overline{X},\overline{Y}),\overline{Z})$.

Corollary 3.10.

(a) $\widehat{S^\diamond}(\overline{X},\overline{Y}) = 0$,

(b) $\widehat{P^\diamond}(\overline{X},\overline{Y}) = \widehat{P}(\overline{X},\overline{Y})$.

(c) $\widehat{R^\diamond}(\overline{X},\overline{Y}) = \widehat{R}(\overline{X},\overline{Y})$.

4. Hashiguchi connection

In this section we establish an intrinsic proof of the existence and uniqueness theorem of the Hashiguchi connection. Moreover, the relationship between this connection and the Cartan connection ∇ is obtained.

Theorem 4.1. *Let (M,L) be a Finsler manifold and g the Finsler metric defined by L. Let D^* be a regular connection on $\pi^{-1}(TM)$ such that*

(a) *D^* is vertically metric*: $D^*_{\gamma\overline{X}}\,g = 0$,

(b) *The (h)hv-torsion tensor T^* of D^* satisfies*: $g(T^*(\overline{X},\overline{Y}),\overline{Z}) = g(T^*(\overline{X},\overline{Z}),\overline{Y})$,

(c) *The (h)h-torsion of D^* vanishes*: $Q^* = 0$,

(d) *The (v)hv-torsion of D^* vanishes*: $\widehat{P^*} = 0$

(e) $D^*_{h^*X}L = 0$, *h^* being the horizontal projector of D^*.*

Then, the nonlinear connection Γ^* associated with the connection D^* coincides with the Barthel connection : $\Gamma = [J,G]$.

To prove this theorem, we need the following lemma:

Lemma 4.2. *Let (M,L) be a Finsler manifold. For a regular connection D^* on $\pi^{-1}(TM)$ satisfying conditions (a) and (b) of Theorem 4.1, the (h)hv-torsion of D^* coincides with the (h)hv-torsion of the Cartan connection: $T^* = T$.*

Proof. The connection D^* being regular, let h^* and v^* be the horizontal and vertical projectors associated with the decomposition (1.3): $h^* = \beta^* \circ \rho, v^* = I - \beta^* \circ \rho$.
As D^* is non-metric with nonzero torsion $\mathbf{T}^*$, then D^* is completely determined by Equation (3.6) with D and $\mathbf{T}$ replaced by D^* and $\mathbf{T}^*$ respectively.

Replacing X, Y, Z by $\gamma\overline{X}, h^*Y, h^*Z$ in (3.6) and using conditions (a) and (b), taking into account the fact that $\rho \circ \gamma = 0$ and $\rho \circ h^* = \rho$, we get

$$2g(D^*_{\gamma\overline{X}}\rho Y, \rho Z) = \gamma\overline{X} \cdot g(\rho Y, \rho Z) + g(\rho Y, \rho[h^*Z, \gamma\overline{X}]) + g(\rho Z, \rho[\gamma\overline{X}, h^*Y]). \quad (4.1)$$

On the other hand, since the difference between two nonlinear connections is a semi-basic vector form on TM [4] and the vertical distribution is completely integrable, then, noting that $\rho(V(TM)) = 0$, we get

$$\rho[h^*Z, \gamma\overline{X}] = \rho[hZ, \gamma\overline{X}], \quad (4.2)$$

h being the horizontal projector of the Cartan connection (or the Barthel connection).

Now, from (4.1) and (4.2), using Theorem 2.4(i), we get

$$D^*_{\gamma\overline{X}}\overline{Y} = \nabla_{\gamma\overline{X}}\overline{Y}.$$

Consequently, again by (4.2), we get $T^*(\overline{X}, \overline{Y}) = T(\overline{X}, \overline{Y})$. □

Proof of theorem 4.1:

By Lemma 4.2, $T^*(\overline{X}, \overline{\eta}) = 0$ as T has the same property. From which, together with Proposition 2.2, it follows that $K^* = \gamma^{-1}$ on $V(TM)$ and the associated nonlinear connection Γ^* is given by $\Gamma^* = \beta^* \circ \rho - \gamma \circ K^*$.

Now, we prove that Γ^* has the following properties:

Γ^* ***is conservative:*** $d_{h^*}E(X) = h^*X \cdot E = LD^*_{h^*X}L = 0$, by condition (e).
Γ^* ***is homogenous :*** one can easily show that

$$[\mathcal{C}, v^*](X) = [\mathcal{C}, \gamma(K^*X)] - \gamma(K^*[\mathcal{C}, v^*X]) - \gamma(K^*[\mathcal{C}, h^*X]).$$

From which, using Lemma 2.3 and condition (d), noting that $\gamma\, o\, K^* = id_{V(TM)}, K^* \circ \beta^* = 0$ and $[\mathcal{C}, v^*X]$ is vertical, we get

$$\begin{aligned} [\mathcal{C}, v^*](X) &= -\gamma(K^*[\mathcal{C}, h^*X]) = -\gamma\, o\, K^*([\gamma\overline{\eta}, \beta^*\rho X]) \\ &= -\gamma\, o\, K\{-\gamma(D^*_{\beta^*\rho X}\overline{\eta}) + \beta^*(D^*_{\gamma\overline{\eta}}\rho X - T(\overline{\eta}, \rho X))\} = 0. \end{aligned}$$

Therefore, $[\mathcal{C}, \Gamma^*] = -2[\mathcal{C}, v^*] = 0$ and Γ^* is thus homogenous.

Γ^* ***is torsion-free:*** By the same argument as in the proof of Theorem 3.1, we have

$$[J, v^*](X, Y) = J[\beta^*\rho X, \beta^*\rho Y] - v^*[\gamma\rho X, \beta^*\rho Y] + v^*[\gamma\rho Y, \beta^*\rho X].$$

From which, together with Lemma 2.3 and condition (c), we obtain

$$\begin{aligned} [J, v^*](X, Y) = &\ J\{\gamma(\widehat{R^*}(\rho X, \rho Y)) + \beta^*(D^*_{h^*X}\rho Y - D^*_{h^*Y}\rho X)\} \\ &-\gamma \circ K^*\{-\gamma(D^*_{h^*Y}\rho X) + \beta^*(D^*_{JX}\rho Y - T(\rho X, \rho Y))\} \\ &+\gamma \circ K^*\{-\gamma(D^*_{h^*X}\rho Y) + \beta^*(D^*_{JY}\rho X - T(\rho Y, \rho X))\} = 0. \end{aligned}$$

Hence $t := \frac{1}{2}[J, \Gamma^*] = -[J, v^*] = 0$.
By the uniqueness of the Barthel connection (Theorem 1.3), Γ^* coincides with the Barthel connection $\Gamma = [J, G]$. □

Theorem 4.3. *Let* (M, L) *be a Finsler manifold and* g *the Finsler metric defined by* L*. There exists a unique regular connection* D^* *on* $\pi^{-1}(TM)$ *such that*

(a) D^* *is vertically metric:* $D^*_{\gamma\overline{X}}\, g = 0$,

(b) *The (h)hv-torsion* T^* *of* D^* *satisfies:* $g(T^*(\overline{X}, \overline{Y}), \overline{Z}) = g(T^*(\overline{X}, \overline{Z}), \overline{Y})$,

(c) *The (h)h-torsion of* D^* *vanishes:* $Q^* = 0$,

(d) *The (v)hv-torsion of* D^* *vanishes:* $\widehat{P^*} = 0$,

(e) $D^*_{h^*X} L = 0$.

Such a connection is called the Hashiguchi connection associated with the Finsler manifold (M, L).

Proof. First we prove the ***uniqueness***. The connection D^* being regular, let Γ^* be its associated nonlinear connection and h^* and v^* its horizontal and vertical projectors. By Theorem 4.1, we have $\Gamma^* = \Gamma = [J, G]$, and consequently

$$v^* = v = \gamma \circ K, \qquad h^* = h = \beta \circ \rho, \qquad K^* = K, \qquad \beta^* = \beta. \tag{4.3}$$

Also, by Lemma 4.2, we have $T^*(KX, \rho Y) = T(KX, \rho Y)$.
Consequently,

$$D^*_{vX}\overline{Y} = \nabla_{vX}\overline{Y}. \tag{4.4}$$

Now, using axiom (d), taking into account (4.3), the definition of P^* and the identities $K \circ J = \rho$ and $K \circ h = 0$, we get

$$\begin{aligned} 0 = \widehat{P}^*(\rho X, \rho Y) &= \mathbf{K}^*(hX, JY)\overline{\eta} = -D^*_{hX} D^*_{JY}\overline{\eta} + D^*_{JY} D^*_{hX}\overline{\eta} + D^*_{[hX, JY]}\overline{\eta} \\ &= -D^*_{hX}\rho Y + K[hX, JY]. \end{aligned}$$

Hence,

$$D^*_{hX}\rho Y = K[hX, JY] = \nabla_{[hX, JY]}\overline{\eta} = \nabla_{hX}\rho Y + \widehat{P}(\rho X, \rho Y). \tag{4.5}$$

Consequently, (4.4) and (4.5) imply that

$$D^*_X\overline{Y} = \nabla_X\overline{Y} + \widehat{P}(\rho X, \overline{Y}), \tag{4.6}$$

which uniquely determines the connection D^*.

To prove the ***existence*** of D^*, we define D^* by (4.6) and prove the following properties:

D^* ***satisfies condition (a)***: From (4.6), as $\rho \circ \gamma = 0$, we get $D^*_{\gamma\overline{X}}\overline{Y} = \nabla_{\gamma\overline{X}}\overline{Y}$. Consequently, $D^*_{\gamma\overline{X}}\, g = \nabla_{\gamma\overline{X}}\, g = 0$, $\forall\ \overline{X} \in \mathfrak{X}(\pi(M))$.

D^* ***satisfies condition (b)***: As $D^*_{\gamma\overline{X}}\rho Y = \nabla_{\gamma\overline{X}}\rho Y$, taking into account (4.2), one concludes that $T^*(\overline{X}, \overline{Y}) = T(\overline{X}, \overline{Y})$. T^* has the property (b) as T does.

D^* ***satisfies conditions (c)***: Setting $\overline{Y} = \overline{\eta}$ into (4.6), taking into account the fact that $\widehat{P}(\overline{X}, \overline{\eta}) = 0$, it follows that $K^* = K$. Since $T^*(\overline{X}, \overline{\eta}) = 0$ (as $T^* = T$), we have $v^* = \gamma \circ K^* = \gamma \circ K = v$ (by Proposition 2.2). Consequently, $h^* = h$ and $\beta^* = \beta$. Hence, from (4.6) and the property that $\widehat{P}$ is symmetric, one gets

$$\begin{aligned} Q^*(\overline{X}, \overline{Y}) &= D^*_{\beta^*\overline{X}}\overline{Y} - D^*_{\beta^*\overline{Y}}\overline{X} - \rho[\beta^*\overline{X}, \beta^*\overline{Y}] = D^*_{\beta\overline{X}}\overline{Y} - D^*_{\beta\overline{Y}}\overline{X} - \rho[\beta\overline{X}, \beta\overline{Y}] \\ &= \nabla_{\beta\overline{X}}\overline{Y} + \widehat{P}(\overline{X}, \overline{Y}) - \nabla_{\beta\overline{Y}}\overline{X} - \widehat{P}(\overline{Y}, \overline{X}) - \rho[\beta\overline{X}, \beta\overline{Y}] = Q(\overline{X}, \overline{Y}) = 0. \end{aligned}$$

D^* ***satisfies condition (d)***: As $K\circ\beta=0$ and $\widehat{P}(\overline{X},\overline{\eta})=0$, using (4.6), we get

$$\begin{aligned}\widehat{P^*}(\overline{X},\overline{Y}) &= P^*(\overline{X},\overline{Y})\overline{\eta}=\mathbf{K}^*(\beta\overline{X},\gamma\overline{Y})\overline{\eta}\\ &= -D^*_{\beta\overline{X}}D^*_{\gamma\overline{Y}}\overline{\eta}+D^*_{\gamma\overline{Y}}D^*_{\beta\overline{X}}\overline{\eta}+D^*_{[\beta\overline{X},\gamma\overline{Y}]}\overline{\eta}=-D^*_{\beta\overline{X}}\nabla_{\gamma\overline{Y}}\overline{\eta}+\nabla_{[\beta\overline{X},\gamma\overline{Y}]}\overline{\eta}\\ &= -\nabla_{\beta\overline{X}}\nabla_{\gamma\overline{Y}}\overline{\eta}-\widehat{P}(\overline{X},\nabla_{\gamma\overline{Y}}\overline{\eta})+\nabla_{[\beta\overline{X},\gamma\overline{Y}]}\overline{\eta}\\ &= \{-\nabla_{\beta\overline{X}}\nabla_{\gamma\overline{Y}}\overline{\eta}+\nabla_{[\beta\overline{X},\gamma\overline{Y}]}\overline{\eta}\}-\widehat{P}(\overline{X},\overline{Y})=0.\end{aligned}$$

D^* ***satisfies condition(e)***: As $h^*=h$, then

$$L\,D^*_{h^*X}L=D^*_{hX}E=hX\cdot E=2g(\nabla_{hX}\overline{\eta},\overline{\eta})=0.$$

This complete the proof. $\square$

It is to be noted that when we localize the above result, the *local expressions* obtained coincide with the classical expressions found in [9], [6], [10]...etc. (c.f. Appendix 2).

In view of the above theorem taking Theorem 2.8 into account, we have:

Theorem 4.4. *The Hashiguchi connection D^* is given in terms of the Cartan connection (or the Berwald connection) by:*

$$D^*_X\overline{Y}=\nabla_X\overline{Y}+\widehat{P}(\rho X,\overline{Y})=D^\circ_X\overline{Y}+T(KX,\overline{Y}).\tag{4.7}$$

In particular, we have

(a) $D^*_{\gamma\overline{X}}\overline{Y}=\nabla_{\gamma\overline{X}}\overline{Y}=D^\circ_{\gamma\overline{X}}\overline{Y}+T(\overline{X},\overline{Y})$.

(b) $D^*_{\beta\overline{X}}\overline{Y}=\nabla_{\beta\overline{X}}\overline{Y}+\widehat{P}(\overline{X},\overline{Y})=D^\circ_{\beta\overline{X}}\overline{Y}$.

Corollary 4.5. *For every $\overline{X},\overline{Y},\overline{Z}\in\mathfrak{X}(\pi(M))$, we have*

(a) $D^*_{\gamma\overline{X}}g=0$,

(b) $(D^*_{\beta\overline{X}}g)(\overline{Y},\overline{Z})=-2g(\widehat{P}(\overline{X},\overline{Y}),\overline{Z})$.

Consequently, (M,L) is a Landsberg manifold if, and only if, $D^*_{\beta\overline{X}}g=0$.

Concerning the curvature tensors of the Hashiguchi connection D^*, we have:

Proposition 4.6. *The curvature tensor $\boldsymbol{K}^*$ of the Hashiguchi connection D^* is given in terms of the curvature tensor $\boldsymbol{K}$ of the Cartan connection ∇ by:*

$$\boldsymbol{K}^*(X,Y)\overline{Z}=\boldsymbol{K}(X,Y)\overline{Z}-\widehat{P}(\boldsymbol{T}(X,Y),\overline{Z})-\mathfrak{U}_{X,Y}\{(\nabla_X\widehat{P})(\rho Y,\overline{Z})+\widehat{P}(\rho X,\widehat{P}(\rho Y,\overline{Z}))\}.$$

In particular, we have

(a) $S^*(\overline{X},\overline{Y})\overline{Z}=S(\overline{X},\overline{Y})\overline{Z}$.

(b) $P^*(\overline{X},\overline{Y})\overline{Z}=P(\overline{X},\overline{Y})\overline{Z}+\widehat{P}(T(\overline{X},\overline{Y}),\overline{Z})+(\nabla_{\gamma\overline{Y}}\widehat{P})(\overline{X},\overline{Z})$.

(c) $R^*(\overline{X},\overline{Y})\overline{Z}=R(\overline{X},\overline{Y})\overline{Z}-\mathfrak{U}_{\overline{X},\overline{Y}}\{(\nabla_{\beta\overline{X}}\widehat{P})(\overline{Y},\overline{Z})+\widehat{P}(\overline{X},\widehat{P}(\overline{Y},\overline{Z}))\}$.

Corollary 4.7.

(a) $\widehat{S^*}(\overline{X},\overline{Y})=0$.

(b) $\widehat{P^*}(\overline{X},\overline{Y})=0,\ \ P^*(\overline{X},\overline{\eta})\overline{Y}=0,\ \ P^*(\overline{\eta},\overline{X})\overline{Y}=-\widehat{P}(\overline{X},\overline{Y})$.

(c) $\widehat{R^*}(\overline{X},\overline{Y})=\widehat{R}(\overline{X},\overline{Y})$.

We terminate the paper by some comments:

- *On a Finsler manifold* (M,L), *there are canonically associated four linear connections: the Cartan connection* ∇ *(Thm. 2.4), the Chern connection* $D^\diamond$ *(Thm. 3.6), the Hashiguchi connection* D^* *(Thm. 4.3) and the Berwald connection* D° *(Thm. 2.6). The nonlinear connection associated with each of these linear connections coincides with the Barthel connection.*

- *There are two methods by which a given Finsler connection is converted to some other connection. From Theorem 3.7, we observe that the Chern connection* $D^\diamond$ *is obtained from the Cartan connection* ∇ *by subtracting the (h)hv-torsion* T *from its vertical counterpart. Moreover, from Theorem 4.4, we observe that the Hashiguchi connection* D^* *is obtained from the Cartan connection by adding the (v)hv-torsion* $\widehat{P}$ *to its horizontal counterpart. The former process is called the* C*-process and the latter process is called the* P^1*-process by M. Matsumoto [9].*

  Interestingly, the two processes commute with each other. If we apply to the Cartan connection the C*-process followed by the* P^1*-process, we obtain the Berwald connection* D° *(passing by the Chern connection) (cf. Theorem 2.8). On the other hand, if we apply the* P^1*-process followed by the* C*-process, we also obtain the Berwald connection (passing by the Hashiguchi connection). That is,*

$$D^\circ \xleftarrow{C\text{-process}} D^* \xleftarrow{P^1\text{-process}} \nabla \xrightarrow{C\text{-process}} D^\diamond \xrightarrow{P^1\text{-process}} D^\circ$$

Appendix 1. Intrinsic Comparison

The following tables establish a concise comparison concerning the canonical linear connections in Finsler geometry as well as the fundamental geometric objects associated with them.

Table 1.

connection	**Cartan:** ∇	**Chern:** $D^{\diamond}$	**Hashiguchi:** D^{*}	**Berwald:** D°
v-counterpart	$\nabla_{\gamma\overline{X}}\overline{Y}$	$D^{\diamond}_{\gamma\overline{X}}\overline{Y} = \nabla_{\gamma\overline{X}}\overline{Y} - T(\overline{X},\overline{Y})$	$D^{*}_{\gamma\overline{X}}\overline{Y} = \nabla_{\gamma\overline{X}}\overline{Y}$	$D^{\circ}_{\gamma\overline{X}}\overline{Y} = \nabla_{\gamma\overline{X}}\overline{Y} - T(\overline{X},\overline{Y})$
h-counterpart	$\nabla_{\beta\overline{X}}\overline{Y}$	$D^{\diamond}_{\beta\overline{X}}\overline{Y} = \nabla_{\beta\overline{X}}\overline{Y}$	$D^{*}_{\beta\overline{X}}\overline{Y} = \nabla_{\beta\overline{X}}\overline{Y} + \widehat{P}(\overline{X},\overline{Y})$	$D^{\circ}_{\beta\overline{X}}\overline{Y} = \nabla_{\beta\overline{X}}\overline{Y} + \widehat{P}(\overline{X},\overline{Y})$
(h)v-torsion	0	0	0	0
(h)hv-torsion	T	0	T	0
(h)h-torsion	0	0	0	0
(v)v-torsion	0	0	0	0
(v)hv-torsion	$\widehat{P} = \nabla_{G}T$	$\widehat{P}$	0	0
(v)h-torsion	$\widehat{R} = -K\mathfrak{R}$	$\widehat{R}$	$\widehat{R}$	$\widehat{R}$
v-curvature	S	0	S	0
hv-curvature	P	$P^{\diamond}$	P^{*}	P°
h-curvature	R	$R^{\diamond}$	R^{*}	R°
v-metricity	$\nabla_{\gamma\overline{X}}g = 0$	$D^{\diamond}_{\gamma\overline{X}}g = 2g(T(\overline{X},.),.)$	$D^{*}_{\gamma\overline{X}}g = 0$	$D^{\circ}_{\gamma\overline{X}}g = 2g(T(\overline{X},.),.)$
h-metricity	$\nabla_{\beta\overline{X}}g = 0$	$D^{\diamond}_{\beta\overline{X}}g = 0$	$D^{*}_{\beta\overline{X}}g = -2g(\widehat{P}(\overline{X},.),.)$	$D^{\circ}_{\beta\overline{X}}g = -2g(\widehat{P}(\overline{X},.),.)$

Table 2.

connection	**curvature tensors**
Cartan	v-curvature: $S(\overline{X},\overline{Y})\overline{Z} := -\nabla_{\gamma\overline{X}}\nabla_{\gamma\overline{Y}}\overline{Z} + \nabla_{\gamma\overline{Y}}\nabla_{\gamma\overline{X}}\overline{Z} + \nabla_{[\gamma\overline{X},\gamma\overline{X}]}\overline{Z}.$ hv-curvature: $P(\overline{X},\overline{Y})\overline{Z} := -\nabla_{\beta\overline{X}}\nabla_{\gamma\overline{Y}}\overline{Z} + \nabla_{\gamma\overline{Y}}\nabla_{\beta\overline{X}}\overline{Z} + \nabla_{[\beta\overline{X},\gamma\overline{X}]}\overline{Z}.$ h-curvature: $R(\overline{X},\overline{Y})\overline{Z} := -\nabla_{\beta\overline{X}}\nabla_{\beta\overline{Y}}\overline{Z} + \nabla_{\beta\overline{Y}}\nabla_{\beta\overline{X}}\overline{Z} + \nabla_{\beta\overline{X},\beta\overline{X}]}\overline{Z}.$
Chern	$S^{\diamond}(\overline{X},\overline{Y})\overline{Z} = 0.$ $P^{\diamond}(\overline{X},\overline{Y})\overline{Z} = P(\overline{X},\overline{Y})\overline{Z} - T(\widehat{P}(\overline{X},\overline{Y}),\overline{Z}) + (\nabla_{\beta\overline{X}}T)(\overline{Y},\overline{Z}).$ $R^{\diamond}(\overline{X},\overline{Y})\overline{Z} = R(\overline{X},\overline{Y})\overline{Z} - T(\widehat{R}(\overline{X},\overline{Y}),\overline{Z}).$
Hashiguchi	$S^{*}(\overline{X},\overline{Y})\overline{Z} = S(\overline{X},\overline{Y})\overline{Z}.$ $P^{*}(\overline{X},\overline{Y})\overline{Z} = P(\overline{X},\overline{Y})\overline{Z} + \widehat{P}(T(\overline{X},\overline{Y}),\overline{Z}) + (\nabla_{\gamma\overline{Y}}\widehat{P})(\overline{X},\overline{Z}).$ $R^{*}(\overline{X},\overline{Y})\overline{Z} = R(\overline{X},\overline{Y})\overline{Z} - \mathfrak{U}_{\overline{X},\overline{Y}}\{(\nabla_{\beta\overline{X}}\widehat{P})(\overline{Y},\overline{Z}) + \widehat{P}(\overline{X},\widehat{P}(\overline{Y},\overline{Z}))\}.$
Berwald	$S^{\circ}(\overline{X},\overline{Y})\overline{Z} = 0.$ $P^{\circ}(\overline{X},\overline{Y})\overline{Z} = P(\overline{X},\overline{Y})\overline{Z} + (\nabla_{\gamma\overline{Y}}\widehat{P})(\overline{X},\overline{Z}) + \widehat{P}(T(\overline{Y},\overline{X}),\overline{Z}) +$ $+ \widehat{P}(\overline{X},T(\overline{Y},\overline{Z})) + (\nabla_{\beta\overline{X}}T)(\overline{Y},\overline{Z}) -$ $- T(\overline{Y},\widehat{P}(\overline{X},\overline{Z})) - T(\widehat{P}(\overline{X},\overline{Y}),\overline{Z}).$ $R^{\circ}(\overline{X},\overline{Y})\overline{Z} = R(\overline{X},\overline{Y})\overline{Z} - T(\widehat{R}(\overline{X},\overline{Y}),\overline{Z}) - \mathfrak{U}_{\overline{X},\overline{Y}}\{(\nabla_{\beta\overline{X}}\widehat{P})(\overline{Y},\overline{Z})$ $+ \widehat{P}(\overline{X},\widehat{P}(\overline{Y},\overline{Z}))\}.$

Appendix 2. Local Comparison

For the sake of completeness, we present in this appendix a brief and concise survey of the local expressions of the most important geometric objects treated in the paper.

Let $(U, (x^i))$ be a system of local coordinates on M and $(\pi^{-1}(U), (x^i, y^i))$ the associated system of local coordinates on TM. We use the following notations:
$(\partial_i) := (\frac{\partial}{\partial x^i})$: the natural basis of T_xM, $x \in M$,
$(\dot{\partial}_i) := (\frac{\partial}{\partial y^i})$: the natural basis of $V_u(\mathcal{T}M)$, $u \in \mathcal{T}M$,
$(\partial_i, \dot{\partial}_i)$: the natural basis of $T_u(\mathcal{T}M)$,
$(\overline{\partial}_i)$: the natural basis of the fiber over u in $\pi^{-1}(TM)$ ($\overline{\partial}_i$ is the lift of ∂_i at u).

To a Finsler manifold (M, L), we associate the geometric objects:
$g_{ij} := \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2 = \dot{\partial}_i\dot{\partial}_j E$: the Finsler metric tensor,
G^h: the components of the canonical spray,
$G^h_i := \dot{\partial}_i\, G^h$,
$G^h_{ij} := \dot{\partial}_j\, G^h_i = \dot{\partial}_j\dot{\partial}_i\, G^h$,
$(\delta_i) := (\partial_i - G^h_i\, \dot{\partial}_h)$: the basis of $H_u(\mathcal{T}M)$ adapted to G^h_i,
$(\delta_i, \dot{\partial}_i)$: the basis of $T_u(\mathcal{T}M) = H_u(\mathcal{T}M) \oplus V_u(\mathcal{T}M)$ adapted to G^h_i.

We have:
$\gamma(\overline{\partial}_i) = \dot{\partial}_i$,
$\rho(\partial_i) = \overline{\partial}_i, \quad \rho(\dot{\partial}_i) = 0, \ \ \rho(\delta_i) = \overline{\partial}_i$,
$\beta(\overline{\partial}_i) = \delta_i$,
$J(\partial_i) = \dot{\partial}_i, \quad J(\dot{\partial}_i) = 0, \ \ J(\delta_i) = \dot{\partial}_i$,
$h := \beta o \rho = dx^i \otimes \partial_i - G^i_j\, dx^j \otimes \dot{\partial}_i, \qquad v := \gamma o K = dy^i \otimes \dot{\partial}_i + G^i_j\, dx^j \otimes \dot{\partial}_i.$

We define:
$\gamma^h_{ij} := \frac{1}{2}\, g^{h\ell}(\partial_i\, g_{\ell j} + \partial_j\, g_{i\ell} - \partial_\ell\, g_{ij})$,
$C^h_{ij} := \frac{1}{2}\, g^{h\ell}(\dot{\partial}_i\, g_{\ell j} + \dot{\partial}_j\, g_{i\ell} - \dot{\partial}_\ell\, g_{ij}) = \frac{1}{2}\, g^{h\ell}\, \dot{\partial}_i\, g_{\ell j}$,
$\Gamma^h_{ij} := \frac{1}{2}\, g^{h\ell}(\delta_i\, g_{\ell j} + \delta_j\, g_{i\ell} - \delta_\ell\, g_{ij})$.

For a Finsler connection $F\Gamma$: $(\ \mathbf{F}^h_{ij}, \mathbf{N}^h_i,\ \mathbf{C}^h_{ij})$, we have two covariant derivatives:
The h-covariant derivative $A^i_{j|k}$ given by $A^i_{j|k} := \delta_k\, A^i_j + A^m_j\, \mathbf{F}^i{}_{mk} - A^i_m\, \mathbf{F}^m_{jk}$,
The v-covariant derivative $A^i_j|_k$ given by $A^i_j|_k := \dot{\partial}_k\, A^i_j + A^m_j\, \mathbf{C}^i_{mk} - A^i_m\, \mathbf{C}^m_{jk}$.

The canonical spray and the canonical connections in Finsler geometry are as follows:

- The canonical spray G: $G^h = \frac{1}{2}\, \gamma^h_{ij}\; y^i\, y^j$,
- The Barthel connection Γ: $G^h_i = \dot{\partial}_i\, G^h$,
- The Cartan connection $C\Gamma$: $(\ \Gamma^h_{ij}, G^h_i, C^h_{ij})$,
- The Chern (Rund) connection $R\Gamma$: $(\ \Gamma^h_{ij}, G^h_i, 0)$,
- The Hashiguchi connection $H\Gamma$: $(\ G^h_{ij}, G^h_i, C^h_{ij})$,
- The Berwald connection $B\Gamma$: $(\ G^h_{ij}, G^h_i,\ 0)$,

where G^h_{ij} is given by: $G^h_{ij} = \dot{\partial}_j\, \dot{\partial}_i\, G^h = \Gamma^h_{ij} + C^h_{ij|k}\, y^k = \Gamma^h_{ij} + C^h_{ij|0}\,;\ C^h_{ij|0} := C^h_{ij|k} y^k$.

In the next table, we give the local expressions for the fundamental tensors associated with the above linear connections.

Table 3: Summary of local expressions

connection	**Cartan**	**Chern**	**Hashiguchi**	**Berwald**
$(\mathbf{F}^h_{ij}, \mathbf{N}^h_i, \mathbf{C}^h_{ij})$	$(\Gamma^h_{ij}, G^h_i, C^h_{ij})$	$(\Gamma^h_{ij}, G^h_i, 0)$	$(G^h_{ij}, G^h_i, C^h_{ij})$	$(G^h_{ij}, G^h_i, 0)$
(h)v-torsion	0	0	0	0
(h)hv-torsion	C^i_{jk}	0	C^i_{jk}	0
(h)h-torsion	0	0	0	0
(v)v-torsion	0	0	0	0
(v)hv-torsion	$P^i_{jk} = C^i_{jk\|0}$	P^i_{jk}	0	0
(v)h-torsion	$R^i_{jk} = \delta_k G^i_j - \delta_j G^i_k$	R^i_{jk}	R^i_{jk}	R^i_{jk}
v-curvature	S^h_{ijk}	0	S^h_{ijk}	0
hv-curvature	P^h_{ijk}	$P^{\diamond h}_{ijk}$	P^{*h}_{ijk}	$P^{\circ h}_{ijk}$
h-curvature	R^h_{ijk}	$R^{\diamond h}_{ijk}$	R^{*h}_{ijk}	$R^{\circ h}_{ijk}$
v-metricity	$g_{ij}\|_k = 0$	$g_{ij}\overset{\diamond}{\|}_k = 2C_{ijk}$	$g_{ij}\overset{*}{\|}_k = 0$	$g_{ij}\overset{\circ}{\|}_k = 2C_{ijk}$
h-metricity	$g_{ij\|k} = 0$	$g_{ij\overset{\diamond}{\|}k} = 0$	$g_{ij\overset{*}{\|}k} = -2P_{ijk}$	$g_{ij\overset{\circ}{\|}k} = -2P_{ijk}$
v-cov. derivative	$A^i_j\|_k$	$A^i_j\overset{\diamond}{\|}_k = \dot{\partial}_k A^i_j$	$A^i_j\overset{*}{\|}_k = A^i_j\|_k$	$A^i_j\overset{\circ}{\|}_k = \dot{\partial}_k A^i_j$
h-cov. derivative	$A^i_{j\|k}$	$A^i_{j\overset{\diamond}{\|}k} = A^i_{j\|k}$	$A^i_{j\overset{*}{\|}k}$	$A^i_{j\overset{\circ}{\|}k} = A^i_{j\overset{*}{\|}k}$

The curvature tensors appeared in the above table are given by:

$R^i_{hjk} = K^i_{hjk} + C^i_{hm} R^m_{jk}, \quad P^i_{hjk} = \dot{\partial}_k \Gamma^i_{hj} - C^i_{hk|j} + C^i_{hm} P^m_{jk}, \quad S^i_{hjk} = \mathfrak{U}_{jk}\{C^m_{hk} C^i_{mj}\},$

$R^{\diamond i}_{hjk} = K^i_{hjk}, \quad P^{\diamond i}_{hjk} = \dot{\partial}_k \Gamma^i_{hj},$

$R^{*i}_{hjk} = \mathfrak{U}_{jk}\{\delta_k G^i_{hj} + G^m_{hj} G^i_{mk}\} + C^i_{hm} R^m_{jk}, \quad P^{*i}_{hjk} = \dot{\partial}_k G^i_{hj} - C^i_{hk\overset{*}{|}j}, \quad S^{*i}_{hjk} = S^i_{hjk},$

$R^{\circ i}_{hjk} = \mathfrak{U}_{jk}\{\delta_k G^i_{hj} + G^m_{hj} G^i_{mk}\}, \quad P^{\circ i}_{hjk} = \dot{\partial}_k G^i_{hj} =: G^i_{hjk},$

where $K^i_{hjk} = \mathfrak{U}_{jk}\{\delta_k \Gamma^i_{hj} + \Gamma^m_{hj} \Gamma^i_{mk}\}$ and $\mathfrak{U}_{jk} A_{jk} := A_{jk} - A_{kj}$.

References

[1] H. AKbar-Zadeh, *Initiation to global Finsler geometry*, Elsevier, 2006.

[2] D. Bao, S. S. Chern, and Z. Shen, *An introuduction to Riemann-Finsler geometry*, Springer-Verlag, Berlin, 2000.

[3] P. Dazord, *Propriétés globales des géodésiques des espaces de Finsler*, Thèse d'Etat, (575) Publ. Dept. Math. Lyon, 1969.

[4] J. Grifone, *Structure présque-tangente et connexions,* I, Ann. Inst. Fourier, Grenoble, 22, 1 (1972), 287–334.

[5] ______, *Structure presque-tangente et connexions,* II, Ann. Inst. Fourier, Grenoble, 22, 3 (1972), 291–338.

[6] M. Hashiguchi: *On conformal transformation of Finsler metrics*, J. Math. Kyoto Univ., 16 (1976), 25-50.

[7] J. Klein and A. Voutier, *Formes extérieures génératrices de sprays*, Ann. Inst. Fourier, Grenoble, 18, 1 (1968), 241–260.

[8] M. Matsumoto, *The theory of Finsler connections*, Publication of the study group of geometry, Vol. 5, Dept. Math. Okayama Univ., 1970.

[9] ______, *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha Press, Otsu, 520 Japan, 1986.

[10] R. Miron and M. Anastasiei, *The geometry of Lagrange spaces: Theory and applications*, Kluwer Acad. Publ., no. 59, 1994.

[11] L. Tamássy, *Area and metrical connections in Finsler spaces, Finsler geometries*, A Meeting of Minds, Kluwer Publ. 2000, Proc. of Coll. on Finsler Spaces, Edmonton, July 1998, 263-280.

[12] A. A. Tamim, *General theory of Finsler spaces with applications to Randers spaces*, Ph. D. Thesis, Cairo University, 1991.

[13] Nabil. L. Youssef, S. H. Abed and A. Soleiman, *Cartan and Berwald connections in the Pullback formalism*, submitted [Arxiv No.: 0707.1320].